Revision 4 (Solutions)

Semester Two Examination

Question/Answer Booklet

MATHEMATICS METHODS UNITS 1 AND 2

Section Two: Calculator-assumed

Student Number: In figures

In words

Teacher name

Time allowed for this section

Reading time before commencing work: ten minutes
Working time for section: Fifty minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet
Formula Sheet (retained from Section One)

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in the WACE examinations

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Section Two: Calculator-assumed **65% (66 Marks)**

This section has **eight (8)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is minutes.

Question 1 **(7 marks)**

(a) A sequence is defined by $T_{n+1} = T_n - 7,\ T_1 = 111$.

(i) Determine T_{20}. (1 mark)

Solution
$111 + (19)(-7) = -22$
Specific behaviours
✓ calculates term

(ii) The sum of the first 40 terms, S_{40}. (1 mark)

Solution
$\frac{40}{2}\big(2(111) + (39)(-7)\big) = -1020$
Specific behaviours
✓ calculates sum

(iii) The value of n that maximises S_n. (2 marks)

Solution
$S_n = \frac{n}{2}\big(2(111) - 7(n-1)\big)$ $\frac{dS}{dn} = -\frac{1}{2}(14n - 229) = 0$ $n = 16.36$ $S_{16} = 936$ $S_{17} = 935$ Hence maximum value is 936 when $n = 16$.
Specific behaviours
✓ Differentiate S_n and equate to 0   ✓ states value of n

(b) A geometric sequence with $T_2 = 87.5$ has a sum to infinity of 800. Determine all possible values of T_1 for this sequence. (3 marks)

Solution
$S_\infty = \frac{a}{1-r} \Rightarrow 800(1-r) = a = \frac{T_2}{r} = \frac{87.5}{r}$ $800(1-r) = \frac{87.5}{r} \Rightarrow r = \frac{1}{8}, \frac{7}{8}$ $T_1 = 87.5 \div \frac{1}{8} = 700 \text{ or } T_1 = 87.5 \div \frac{7}{8} = 100$
Specific behaviours
✓ adjusts sum to infinity formula for T_2 ✓ substitutes and solves   ✓ calculates both values of T_1

Question 2 **(8 marks)**

(a) Two students are to be chosen from a class of 18.

(i) Determine how many different pairs of students may be chosen. (1 mark)

Solution
$\binom{18}{2} = 153$
Specific behaviours
✓ calculates combinations

(ii) One of the students in the class is the oldest in the school. What is the probability that this student is included in the pair chosen? (2 marks)

Solution
$\binom{1}{1} \times \binom{17}{1} = 17$ $\frac{17}{153} = \frac{1}{9}$
Specific behaviours
✓ calculates ways to choose oldest ✓ calculates probability

(b) A box contains 13 cans of soup, four of which have tomato as an ingredient and the remainder that do not. Four cans are to be selected at random from the box.

(i) Calculate how many different selections of four cans can be made from the box. (1 mark)

Solution
$\binom{13}{4} = 715$
Specific behaviours
✓ calculates combinations

(ii) Determine the probability that none of the four cans will have tomato as an ingredient. (2 marks)

Solution
$\binom{9}{4} = 126; \frac{126}{715} = 0.176$
Specific behaviours
✓ chooses four from the 9 non-tomato ✓ calculates probability

(iii) Determine the probability that in the selection of four cans, there will be an equal number of cans with and without tomato as an ingredient. (2 marks)

Solution
$\binom{4}{2}\binom{9}{2} = 6 \times 36 = 216; \frac{216}{715} = 0.302$
Specific behaviours
✓ chooses 2 tomato and 2 non-tomato ✓ calculates probability

Question 3 **(8 marks)**

The imprisonment rate R, in number of prisoners per 100 000 people, in the US between the years 1960 and 2000, can be modelled by the following equation, where n is the year.

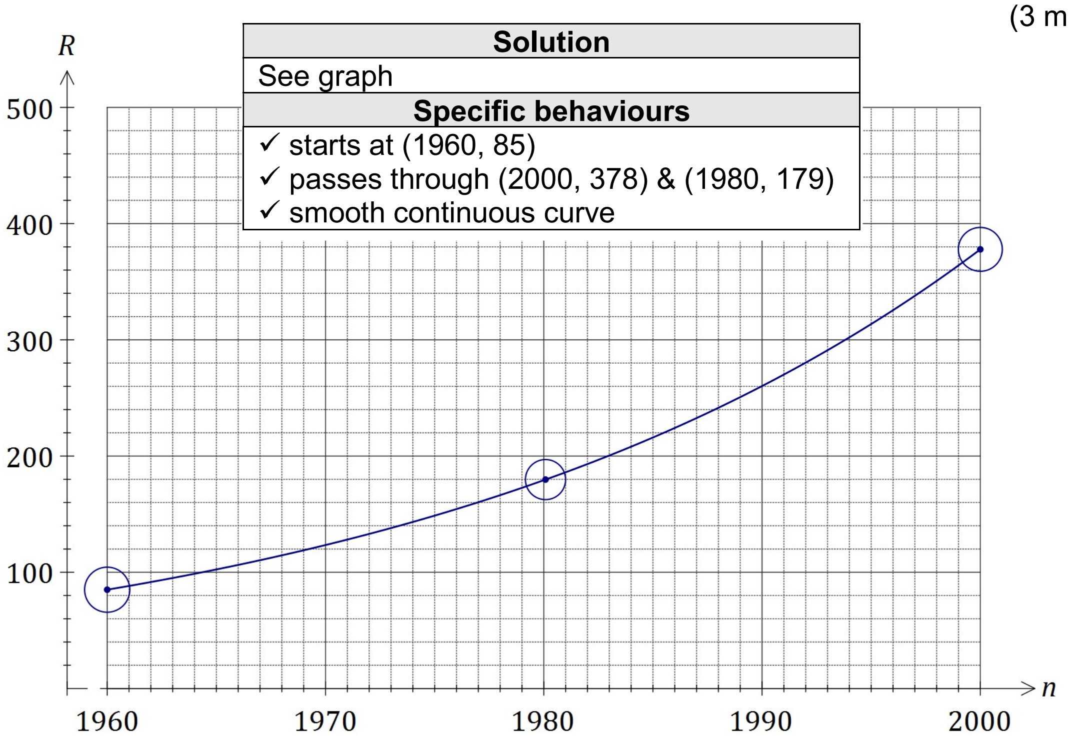
$$R = 85(1.038)^{n-1960}$$

(a) Calculate the imprisonment rate in the year 2000. (1 mark)

Solution
$R = 378$
Specific behaviours
✓ calculates rate

(b) Draw the graph of the imprisonment rate for $1960 \le n \le 2000$ on the axes below. (3 marks)

Solution
See graph
Specific behaviours
✓ starts at (1960, 85)
✓ passes through (2000, 378) & (1980, 179)
✓ smooth continuous curve



(c) The population of the US was 266 million in 1995. Determine the number of prisoners in the US at this time, to the nearest 1 000. (3 marks)

Solution
$R(1995) = 313.564$ $313.564 \div 100000 \times 266000000 = 834080$ $834\,000$ prisoners
Specific behaviours
✓ calculates rate, maintaining accuracy
✓ adjusts for population
✓ rounds at end as required

(d) When R first exceeded 500, steps were taken to address the exponential growth in the prison population and the model no longer applied. In what year did this occur? (1 mark)

Solution
$500 = 85(1.038)^{n-1960} \Rightarrow n = 2007.5 \Rightarrow$ During 2007
Specific behaviours
✓ solves for n

Question 4 **(9 marks)**

The function f is given by $f(x) = x^3 - 3x + 2$.

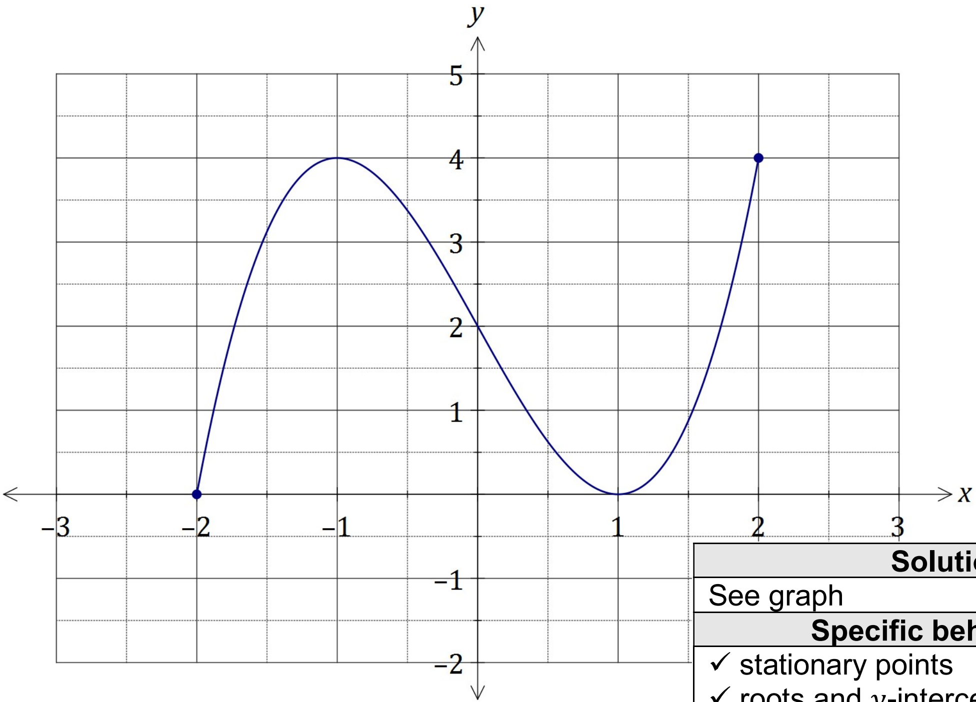
(a) Show that the graph of $y = f(x)$ has two roots and state their coordinates. (2 marks)

Solution
$x^3 - 3x + 2 = (x+2)(x-1)^2$ $(x+2)(x-1)^2 = 0 \Rightarrow x = -2, 1$ Two roots at $(-2, 0)$ and $(1, 0)$
Specific behaviours
✓ factorises ✓ states coordinates of roots

(b) Use calculus techniques to determine the coordinates of all stationary points of the graph of $y = f(x)$. (4 marks)

Solution
$f'(x) = 3x^2 - 3$ $3(x-1)(x+1) = 0 \Rightarrow x = -1, 1$ At $(-1, 4)$and $(1, 0)$
Specific behaviours
✓ differentiates f ✓ shows that $f'(x) = 0$ ✓ solves $f'(x) = 0$ ✓ states coordinates

(c) Sketch the graph of $y = f(x)$ on the axes below for $-2 \le x \le 2$. (3 marks)



Solution
See graph
Specific behaviours
✓ stationary points ✓ roots and y-intercept ✓ smooth curve within given domain

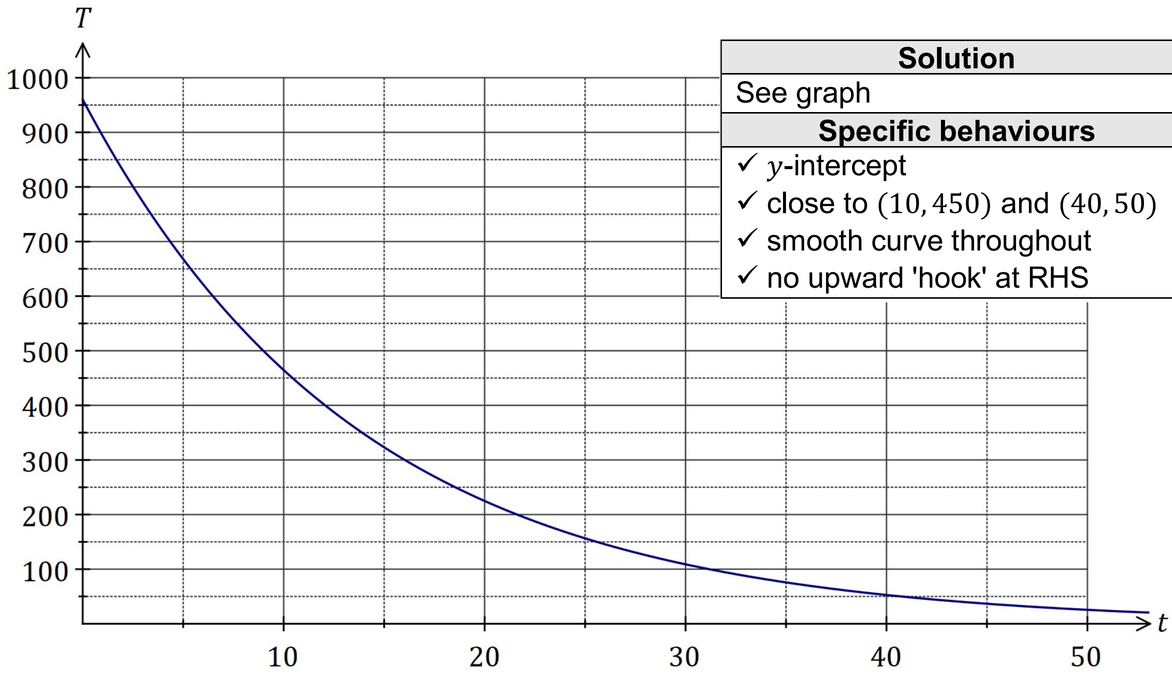
Question 5 **(9 marks)**

The temperature T of a cast taken out of an oven cools according to the model $T = 960(0.93)^t$, where t is the time in minutes since the cast was removed from the oven. T is measured in $°C$.

(a) Determine the fall in temperature of the cast during the first 3 minutes. (2 marks)

Solution
$T = 960(0.93)^3 \approx 772°C$ $\Delta T = 960 - 772 = 188°C$
Specific behaviours
✓ value of T when $t = 3$ ✓ correct drop

(b) Graph the temperature of the cast against time on the axes below. (4 marks)



Solution
See graph
Specific behaviours
✓ y-intercept ✓ close to $(10, 450)$ and $(40, 50)$ ✓ smooth curve throughout ✓ no upward 'hook' at RHS

(c) State the name of this type of function. (1 mark)

Solution
Exponential.
Specific behaviours
✓ correct name

(d) The temperature of the cast falls to room temperature of $14°C$.

(i) Determine the time taken for the cast to reach room temperature. (1 mark)

Solution
$960(0.93)^t = 14 \Rightarrow t = 58.3$ m
Specific behaviours
✓ correct time

(ii) Comment on the usefulness of the model for large values of t. (1 mark)

Solution
For large values of t the model shows that $T \to 0$ but the temperature of the cast only falls to $14°C$ and so model not valid for large T.
Specific behaviours
✓ states not valid, with reason

Question 6 **(8 marks)**

(a) Calculate the area of the minor segment that subtends an arc of $150°$ in a circle of diameter 190 cm. (2 marks)

Solution
$150° = \frac{5\pi}{6}, \qquad r = \frac{190}{2} = 95$ $A = \frac{1}{2}(95)^2\left(\frac{5\pi}{6} - \sin\frac{5\pi}{6}\right) \approx 9\,557 \text{ cm}^2$
Specific behaviours
✓ converts angle, uses correct radius   ✓ calculates area

(b) A chord of length 33 cm subtends an angle of $\frac{\pi}{11}$ at the centre of a circle. Calculate the radius of the circle. (2 marks)

Solution
$33 = 2r\sin\left(\frac{1}{2} \times \frac{\pi}{11}\right)$ $r \approx 116 \text{ cm}$
Specific behaviours
✓ substitutes into formula   ✓ calculates radius

(c) Parallelogram $PQRS$ has side $PQ = 35$ cm, side $QR = 18$ cm and an area of 200 cm^2. Determine the lengths of the diagonals of $PQRS$. (4 marks)

Solution
$\frac{1}{2}(35)(18)\sin x = \frac{200}{2}$ $x = 18.51°, 161.49°$ $L_1 = \sqrt{35^2 + 18^2 - 2(35)(18)\cos 18.51}$ $\approx 18.8 \text{ cm}$ $L_2 = \sqrt{35^2 + 18^2 - 2(35)(18)\cos 161.49}$ $\approx 52.4 \text{ cm}$
Specific behaviours
✓ equation for half area   ✓ both angles of parallelogram   ✓ correct length of one diagonal   ✓ second correct length

Question 7 **(8 marks)**

A council took a random sample of 125 and 172 properties from suburbs P and Q respectively. A total of 36 of the properties in the sample were in arrears with their rates, and 21 of these properties were in suburb Q. 'In arrears' means that payment of rates is overdue.

(a) Council officers wanted to choose 4 of the properties that were in arrears. How many different selections of properties are possible? (2 marks)

Solution
$\binom{36}{4} = 58\,905$
Specific behaviours
✓ indicates use of nCr formula ✓ correct number

(b) Determine the probability that one randomly chosen property from the sample

(i) is not in arrears and is in suburb Q. (2 marks)

Solution
$P = \dfrac{172-21}{125+172} = \dfrac{151}{297} \quad (\approx 0.508)$
Specific behaviours
✓ numerator ✓ denominator

(ii) is in suburb P given that it is in arrears. (1 mark)

Solution
$P = \dfrac{36-21}{36} = \dfrac{15}{36} = \dfrac{5}{12} \quad (= 0.41\bar{6})$
Specific behaviours
✓ correct probability

(c) Justifying your answer with conditional probabilities, comment on whether being in arrears with rates is independent of the suburb the property is in. (3 marks)

Solution
$P(\text{Arrears}\mid P) = \dfrac{15}{125} = 12.0\%$ $P(\text{Arrears}\mid \text{Q}) = \dfrac{21}{172} \approx 12.2\%$ Hence being in arrears is independent of suburb, as conditional probabilities are very similar.
Specific behaviours
✓ calculates $P(\text{Arrears}\mid P)$ ✓ calculates $P(\text{Arrears}\mid Q)$ ✓ correct conclusion

Question 8 **(8 marks)**

A pyramid with a rectangular base of length L and width w has perpendicular height h. The length of the base is five times its width and the sum of the width, length and height is 117 cm.

(a) Calculate the length, height and volume of the pyramid when $w = 15$ cm. (2 marks)

Solution
$L = 5 \times 15 = 75, \qquad h = 117 - 15 - 75 = 27$ $V = \frac{1}{3}(15 \times 75) \times 27 = 10\,125 \text{ cm}^3$
Specific behaviours
✓ correct length and height   ✓ correct volume

(b) Show that the volume of the pyramid is given by $V = 195w^2 - 10w^3$. (2 marks)

Solution
$L = 5w, \qquad h = 117 - w - 5w = 117 - 6w$ $V = \frac{1}{3}(w \times 5w)(117 - 6w)$ $= 195w^2 - 10w^3$
Specific behaviours
✓ expressions for length and height   ✓ substitutes width, length and height correctly

(c) Use calculus to determine the maximum volume of the pyramid and state the dimensions required to achieve this volume. (4 marks)

Solution
$\frac{dV}{dw} = 390w - 30w^2$ $390w - 30w^2 = 0 \Rightarrow w = 0, 13$ $V_{max} = 195(13)^2 - 10(13)^3 = 10\,985 \text{ cm}^3$ $w = 13 \text{ cm}, \qquad L = 65 \text{ cm}, \qquad h = 39 \text{ cm}$
Specific behaviours
✓ correct derivative using given variables   ✓ solves derivative equal to zero   ✓ correct maximum volume   ✓ correct dimensions